Calculus in the AP Physics C Course

The Integral

First Things First: The Antiderivative (Also Known as the Indefinite Integral)

In the simplest case, the *antiderivative* of a function is found by going backward from the derivative of that function.

Example 1*:* Suppose the position of an object is described by the position $x(t) = 2t^3 + 3t^2 + t + 2$

Then the velocity of the object at any time (represented as v(t), here) can be found by taking the derivative of the position, *x*, with respect to time, *t*:

$$v(t) = 6t^2 + 6t + 1$$

Then the acceleration of the object at any time can be found by taking the derivative of the velocity, *v(t)* with respect to time, *t*:

$$a(t) = 12t + 6$$

If *v(t)* is the *derivative* of *x*, then we can say that *x(t)* is the *antiderivative* of *v(t)*. Likewise, if *a(t)* is the derivative of *v(t)*, then *v(t)* is the antiderivative of *a(t).*

Example 2*:* Suppose the velocity of a car is given at any time to be $v(t) = t^2 + 4t + 2$

Find the position *x* of the car at $t = 3$.

To find the position from the velocity function we need to take the antiderivative. That is, what is the function whose derivative is the given velocity function? We find that the antiderivative of v(t) is the position x(t):

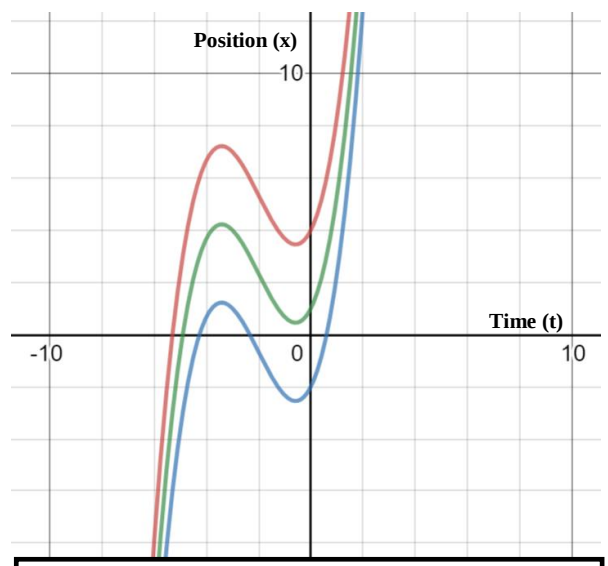
$$x(t) = \frac{t^3}{3} + 2t^2 + 2t + C$$

You can check the answer by simply taking the derivative of *x* and compare it to the velocity function.

Note that a constant C appears in the antiderivative. This is because the derivative of a constant is zero, which means there could be a nonzero constant in the position function, *which would represent the initial position of the car at* $t = 0$. In order to find out what that constant is we would **need more information** given to us about the initial conditions of the problem. For example, if, we know that the initial position of the object that the above equation describes is 5.0 meters, then, we see that the C in the equation must be 5.0 meters, when t = 0.

In general, the antiderivative, f(x), for a power function $f'(x) = ax^n$ may be found using:

$$\boxed{f(x) = \frac{a}{n+1} x^{n+1}}$$



This image shows a few functions that all have the same derivative, $v(t) = 4t + t^2 + 2$. Notice that the slope of each of these functions is the same for every x value.

Example 3: For an object undergoing constant acceleration a = 5 m/s^2, find its velocity as a function of time and its position as a function of time. Its initial velocity, v_0, is 2 m/s, and its initial position is x_0 = -10 meters.

We begin with constant acceleration, and simply take the antiderivative with respect to time to find the velocity as a function of time:

$$a(t) = 5$$

$$v(t) = 5t + C$$

where the constant C in this case must be the initial velocity, v_0, which was given as 2.0 m/s . We have then

$$v(t) = 5t + 2$$

Which, by inspection can be written more generically as:

$$v(t) = at + v_0$$ *This is a kinematic equation, by the way.

In the same way, we find the position as a function of time by taking the antiderivative of the velocity function with respect to time, we get x(t):

$$x(t) = \frac{5}{2}t^2 + 2t + C$$

where the constant here is the initial position of the object, s_o.

$$x(t) = \frac{5}{2}t^2 + 2t - 10$$

Again, this may be written more generically as:

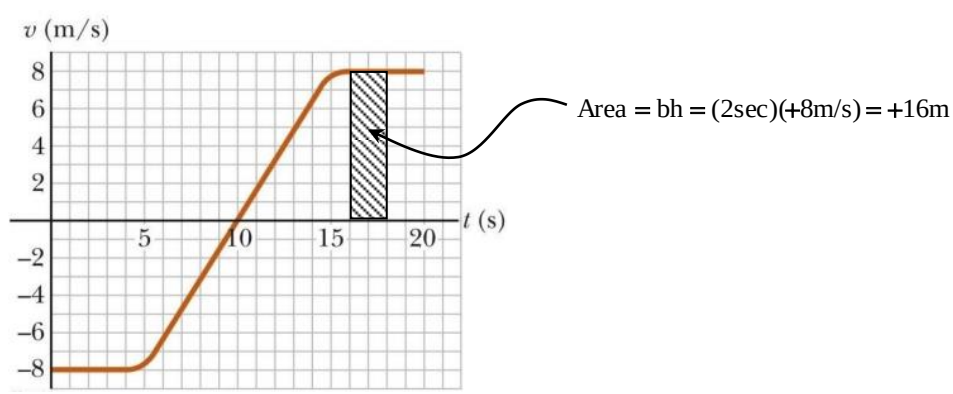
$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$ **This is also a kinematic equation…

The Area-Integral Connection

Recall that the displacement (Δx) for a given time interval, Δt, is the product of the average velocity of an object and the length of the time interval. Symbolically,

$$\Delta x = \bar{v}(\Delta t)$$

Consider the displacement that occurs from t = 16 to 18 seconds on the graph below. This displacement is the area between the plot and the horizontal axis, as shown below:



For shapes that can easily be reduced to triangles and rectangles, the area beneath the “curve” can be easy to evaluate, but suppose we have a position that varies with time (Figure. A).

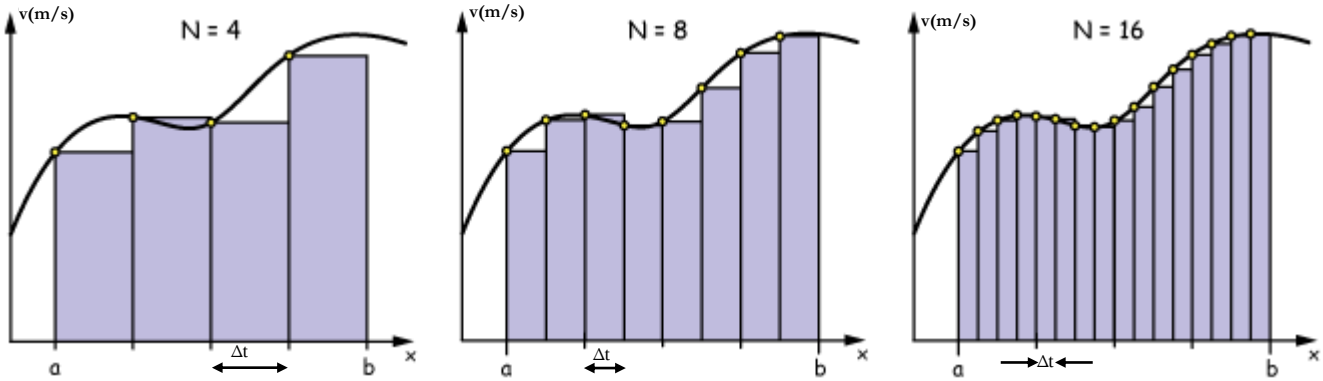


Figure A ***Figure B*** ***Figure C***

To estimate the displacement (i.e., the area under the velocity vs. time graph) for some time interval t = a to b, we divide the area into rectangles of width Δt and height $\boldsymbol{v(t)}$ (Figures A, B, and C).

An estimation for the incremental displacement that occurs during any time Δt is:

$$\Delta x = v(\Delta t)$$

So, to approximate the total area under this curve (and thus the displacement), we can simply sum up the little areas of all the rectangles (numbered from i = 1 to n) from ***a*** to ***b***.

$$\text{Total } \Delta x = \sum_{i=1}^{n} \Delta x = \sum_{i=1}^{n} v(t_i)\Delta t_i \qquad \text{where } v(t_i)(\Delta t_i) \text{ is the area of rectangle i}$$

The notation here indicates that we are adding together the areas for n number of rectangles. Note that n = 4 in Figure A, 8 in Figure B, and 16 in Figure C.

How can we get a more accurate approximation of the area under the curve, which in this case, is the displacement for the object from time ***a*** to time ***b***)? With a little inspection, you probably recognize that if we increase the number of rectangles (n), we will improve our accuracy.

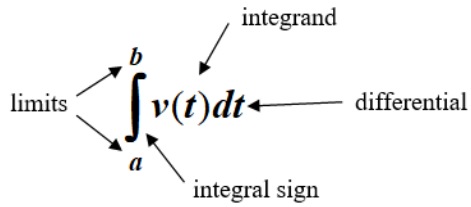
Let's break the area up into more rectangles having smaller widths (smaller Δt's) so that the heights of the rectangles will better fit our curve. Even if we choose a large number of narrow rectangles, we will only get an approximation for the area beneath the curve. But, if we, take the limit for the sum of the areas of these rectangles, as their widths become infinitesimally small, i.e., let Δt → 0, we can find the exact area under the curve. We write:

$$\text{Exact } \Delta x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} v(t_i)dt_i = \int_a^b v(t)dt$$

The limit of this sum is the *integral*, representing the exact area under the *v(t)* vs *t* curve.

Note that the Σ becomes an ∫, the integral symbol which takes the shape of an elongated "S", for “summa”, the Latin word for "sum". Also note that the Δt has become a *dt*, called a differential. The differential *dt* is an infinitesimally small amount of *t* (infinitesimally small ≈ a point in time, in this case), and always accompanies the integral sign. This tells us that we are summing (∫) an infinite number of infinitesimally thick rectangles (having height v(t) and thickness dt) to get the exact whole.

So, let's look at this new notation:



and is read "the *integral of v of t, with respect to t*". The variable in the integrand *v(t)* and the differential variable *dt* **must match** the variable within the integrand before the integration can be carried out. The limits of the integration are from **a** to **b**, because we are looking for the area underneath the graph from t = a to t = b.

Reminder: We're Building to the Area-Integral Connection
But how do we carry out summing by integration? In its simplest form, the *integral* is an *antiderivative – an indefinite integral.* Let's look at the antiderivative again.

Example 3: The velocity of an object varies with time as described by the function v(t) below:

$$v(t) = 6t^2 + 6t + 1$$

Let's say that we wanted to solve for the ***displacement*** that occurs from t = 5 seconds to 6 seconds. We can solve for the antiderivative of this function (as we did previously), which gives us position as a function of time, x(t).

The antiderivative of this function is: $$x(t) = \frac{t^3}{3} + 2t^2 + 2t + C$$

If we use this function to solve for the positions at t = 5 and t = 6, then we can solve for the displacement using:

$$\Delta x = x(6) - x(5)$$

$$\left.\begin{aligned} x(6) &= \frac{6^3}{3} + 2(6)^2 + 2(6) + C = 156 \text{ meters} + C \\ x(5) &= \frac{5^3}{3} + 2(5)^2 + 2(5) + C = 102 \text{ meters} + C \end{aligned}\right\}$$

Some folks get worried right about now, when they see that we don't actually know the positions x(6) or x(5) because there is a constant, C, in these expressions. Relax, it's okay.*

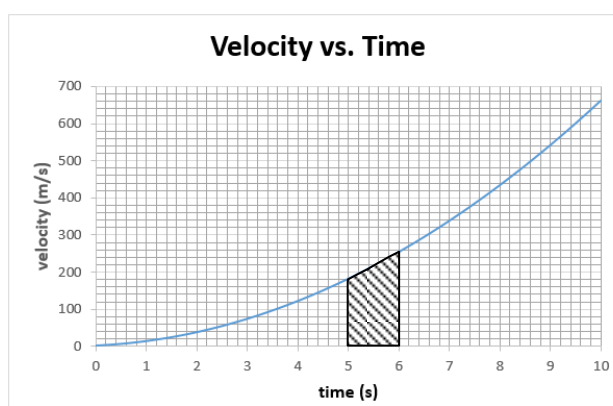
And therefore, the displacement is:

$$\Delta x = 156 \text{ m} + C - (102 \text{ m} + C) = 156 \text{ m} + C - 102 \text{ m} - C = 156 \text{ m} - 102 \text{ m} = 54 \text{ m}$$

The C's cancel out… they ALWAYS will when you evaluate a definite integral.

And what did we really just do? Drumroll, please.

We evaluated the antiderivative at the upper and lower limits (6 second and 5 seconds, respectively), and subtracted the evaluation at the lower limit from that at the upper limit – and THAT is equal to the area underneath the velocity vs. time graph, from 5 to 6 seconds.



The notation for what we just did is:

$$\Delta x = \int_5^6 v(t)dt$$

*Remember that this C is a constant for the function that has physical significance. C, in this function, is the initial position.

The **BIG IDEA** here is this called the Fundamental Theorem of Calculus: if you are trying to find the area beneath a curve for a given function *f*(x) between the boundaries of a and b, that area is equal to the antiderivative F(x) evaluated at the upper boundary (b) minus the antiderivative evaluated at the lower boundary (a).

Fundamental Theorem of Calculus: If the function, *f*, is continuous on [a,b], and F is an antiderivative of *f*; then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Let's play with this a little more before you're off on your own:

Example 4: The velocity of a particle varies as a function of time, such that v(t) = 3t – 2. Solve for the displacement of this particle that occurs from t = 2 to 5 seconds:

$$v(t) = 3t - 2$$

$$\Delta x = \int_2^5 (4t - 2)dt = \left[\frac{4t^2}{2} - 2t\right]_2^5 = \left[2t^2 - 2t\right]_2^5 = \underbrace{2(5)^2 - 2(5)}_{\text{antiderivative evaluated at 5 seconds}} - \underbrace{\left(2(2)^2 - 2(2)\right)}_{\text{antiderivative evaluated at 2 seconds}} = 36\text{ m}$$

Table of Integrals You Need to Know for AP Physics

1. $\int k dx = kx + C$

2. $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$

3. $\int \frac{dx}{x} = \ln|x| + C$

4. $\int \sin x dx = -\cos x + C$

5. $\int \cos x dx = \sin x + C$

6. $\int e^x dx = e^x + C$

Integration Practice

For problems 1-5, evaluate the indefinite integral (antiderivative). For problems 6-10 evaluate the definite integrals. Complete the assignment on a separate piece of paper. $\int 3x^4 dx$

1. $\int (3-2t-t^2)dt$

2. $\int \left[\frac{2}{x^3}+\frac{3}{x^2}+5\right]dx$

3. $\int \left[2x-\frac{1}{2x}\right]dx$

4. $\int t^2(4-t^2)^3 dt$

5. $\int_2^5 9dy$

6. $\int_1^8 r^2 dr$

7. $\int_{0.5}^4 (x^3-6x^2+9x+1)dx$ Graph this problem only, and indicate the area that this definite integral represents.

8. $\int_1^3 \frac{dy}{y}$

9. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 3\sin(2\theta)d\theta$